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Generating functions for permutation representations

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Abstract

We discuss the categorical approach to representations in wreath products, and generalize the Wohlfahrt formula of the exponential generating function for the number of permutation representations.

Keywords: Wreath product; Generating function; Permutation representation

1. Introduction

Suppose that a group A contains only a finite number of subgroups of index d for each positive integer d. As an instance, every finitely generated group satisfies such a property. By the hypothesis, $|\operatorname{Hom}(A,H)| := \sharp\operatorname{Hom}(A,H) < \infty$ for an arbitrary finite group H, where $\operatorname{Hom}(A,H)$ denotes the set of homomorphisms from A to H. Given a sequence $K_0, K_1, K_2, \ldots$ of finite groups K_n such that the first term is the group consisting of only the identity ε, we call the exponential generating function $\sum_{n=0}^{\infty} |\operatorname{Hom}(A,K_n)| t^n/n!$ the *Wohlfahrt series* for K_n. A typical example of such a formal power series comes from the sequence of the symmetric group S_n on $[n] := \{1,2,\ldots,n\}$. In the paper [9], Wohlfahrt proved that

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$$\sum_{n=0}^{\infty}\frac{|\operatorname{Hom}(A,S_n)|}{n!}t^n=\exp\biggl(\sum_{B\leqslant_f A}\frac{1}{|A:B|}t^{|A:B|}\biggr), \tag{1}$$

where the sum $\sum_{B\leqslant_f A}$ is over all subgroups B of A of finite index $|A:B|$.

Throughout the paper, G is a finite group, A_n is the alternating group on $[n]$, $W(D_n)$ is the Weyl group of type D_n, and $G\wr S_n$ and $G\wr A_n$ are the wreath products of G with S_n and A_n, respectively. The exponential formula of the Wohlfahrt series for $G\wr S_n$ is given in the recent papers [4,8] (cf. Corollary 1). It is also shown in the papers [2,6] when A is a finite cyclic group. If K_n is either $G\wr A_n$ or $W(D_n)$, the exponential generating function for the number of solutions in K_n to the equation $x^d=\varepsilon$ was found by Chigira [2]; see also [4].

Recently, Yoshida has developed the theory of generating functions from the categorical point of view, and has presented many applications [10]. In this paper, we carry out investigations into applications of the categorical theory.

We refer to the paper [10] for the notation and terminology on categories. Let $\mathscr{E}$ be a category with any finite coproducts. A connected object J of $\mathscr{E}$ is a noninitial object satisfying the condition that $J=A+B$ implies that A or B is an initial object [10, 5.5]. The category $\mathscr{E}$ is a KS-category if the following condition holds [10, 5.5]:

KS Property. *Any object X is isomorphic to a coproduct of some finite number of connected objects and a coproduct decomposition of X is unique in the following sense: if $X=I_1+\cdots+I_m=J_1+\cdots+J_n$ are two coproduct decompositions into connected objects with canonical injections $i_\alpha : I_\alpha\to X$ and $j_\beta : J_\beta\to X$, then $m=n$ and there exist a permutation $\pi\in S_n$ and isomorphisms $f_\alpha : I_\alpha\to J_{\pi(\alpha)}$ such that $j_{\pi(\alpha)}f_\alpha=i_\alpha$ for all $\alpha=1,\dots,n$.*

A category is skeletally small if a full subcategory consisting of complete representatives of the isomorphism classes of objects is equivalent to a small category [10, 2.1]. Suppose that $\mathscr{E}$ is a skeletally small KS-category. Let $\mathbb{Q}[\![\mathscr{E}^{\mathrm{op}}/\cong]\!]$ be the $\mathbb{Q}$-module of formal power series with exponents in $\mathscr{E}$: $f(t)=\sum'_{X\in\mathscr{E}}a_Xt^X$, $a_X\in\mathbb{Q}$, the summation $\sum'_{X\in\mathscr{E}}$ being over all isomorphism classes of objects of $\mathscr{E}$; the variables t^X is considered as the isomorphism class containing X in the dual category $\mathscr{E}^{\mathrm{op}}$, and hence $t^X=t^Y$ if $X\cong Y$ [10, 4.1]. By the product operation $t^X\cdot t^Y=t^{X+Y}$, $t^{\emptyset}=1$, where $\emptyset$ is an initial object of $\mathscr{E}$, $\mathbb{Q}[\![\mathscr{E}^{\mathrm{op}}/\cong]\!]$ becomes a $\mathbb{Q}$-algebra [10, 5.1, 5.2]. A category is locally finite if every hom set $\operatorname{Hom}(X,Y)$ is a finite set [10, 2.1]. If $\mathscr{E}$ is locally finite, then the generating function of $\mathscr{E}$ is

$$\mathscr{E}(t):=\sum_{X\in\mathscr{E}}{}'\frac{t^X}{|\operatorname{Aut}(X)|}\in\mathbb{Q}[\![\mathscr{E}^{\mathrm{op}}/\cong]\!]$$

[10, 4.2]. According to [10, 6.4], the Wohlfahrt formula (1) is verified on the basis of a categorical result, namely,

[10, 5.8. Theorem]. *Let $\mathscr{E}$ be a skeletally small KS category and let $\mathscr{J} := \mathrm{Con}(\mathscr{E})$ be the full subcategory of connected objects of $\mathscr{E}$. If $\mathscr{E}$ is locally finite, then*

$$\mathscr{E}(t) = \exp\bigl(\mathscr{J}(t)\bigr).$$

Here the power series $\mathscr{J}(t)$ is viewed as an element of $\mathbb{Q}[\![\mathscr{E}^{\mathrm{op}}/\cong]\!]$ through the canonical embedding $\mathscr{J} \subseteq \mathscr{E}$.

In Section 3, we apply [10, 5.8. Theorem] to a certain category related to (A, G)-bisets, and obtain the principle of enumerating the homomorphisms from A to $G \wr S_n$ (cf. Proposition 5). Further, we can find various Wohlfahrt series in Sections 4 and 5. We give the notation and an outline of the results.

Notation.

(a) Let B be a subgroup of A. We denote by A/B the left A-set consisting of all left cosets of B in A with the action given by $a.xB = axB$ for all $a, x \in A$. Define $\mathrm{sgn}_B : A \to \{1, -1\}$ by $\mathrm{sgn}_B(a) = 1$ if a is an even permutation on A/B, and $\mathrm{sgn}_B(a) = -1$ otherwise. It is evident that sgn_B is a homomorphism.

(b) The letter p always stands for a prime. Let ω be a primitive pth root of 1 and $\langle\omega\rangle$ a cyclic group generated by ω. We denote by $\Phi_p(A)$ the intersection of all kernels of homomorphisms from A to $\langle\omega\rangle$. Then $\Phi_p(A)$ is a normal subgroup of A of finite index, and the factor group $A/\Phi_p(A)$ is an elementary abelian p-group. Also, $\Phi_p(A)$ is contained in the kernel of any homomorphism from A to an elementary abelian p-group. Let $\mathscr{C}_p(A)$ denote the set of minimal subgroups of $A/\Phi_p(A)$. Each element of $\mathscr{C}_p(A)$ is a cyclic group of order p, and is of the form $\langle\bar{c}\rangle$ for an element $c \in A - \Phi_p(A)$ with $c^p \in \Phi_p(A)$, where $\bar{c}$ denotes the left coset $c\,\Phi_p(A)$ of $\Phi_p(A)$ in A.

(c) Let B be a subgroup of A of index n, and let $T_B = \{a_1, a_2, \ldots, a_n\}$ be a left transversal of B. The transfer $V_{B/\Phi_p(B)}$ from A to $B/\Phi_p(B)$ is defined by

$$V_{B/\Phi_p(B)}(a) := \prod_{j=1}^{n} a_{j'}^{-1} a a_j \Phi_p(B), \quad \text{where } aa_jB = a_{j'}B,$$

for all $a \in A$. It is well known that $V_{B/\Phi_p(B)}$ is independent of the choice of T_B and is a homomorphism.

If a sequence $\chi_1, \chi_2, \ldots$ of homomorphisms $\chi_n \in \mathrm{Hom}(G \wr S_n, \langle\omega\rangle)$, $n = 1, 2, \ldots$, satisfies a certain condition, then the Wohlfahrt series for $\mathrm{Ker}\,\chi_n$ is described by a summation of exponential formulas to which sgn_B and $V_{B/\Phi_p(B)}$ are closely related (cf. Theorems 1–3). In particular, we can present the Wohlfahrt series for $G \wr A_n$ (cf. Corollary 2). As for $W(D_n)$, the result can be considered in a generalized situation (cf. Corollary 3).

2. A categorical view of Hom$(A, G \wr S_n)$

We start with the definition of the wreath product of two groups (see, e.g., [3, Chapter I, 2.1]). For each finite set X, let G^X be the set of mappings from X to G. The wreath product $G \wr H$ of G with a permutation group H on a finite set X is the cartesian product $G^X \times H$ with the composition law

$$(f;\pi)(f_*;\pi_*) := \bigl(f \cdot \bigl(f_* \circ \pi^{-1}\bigr); \pi\pi_*\bigr), \quad (f;\pi), (f_*;\pi_*) \in G^X \times H,$$

where

$$\bigl(f \cdot \bigl(f_* \circ \pi^{-1}\bigr)\bigr)(x) = f(x) f_*\bigl(\pi^{-1}(x)\bigr)$$

for all $x \in X$. In particular, $G \wr S_n$ is the wreath product of G with a symmetric group S_n on $[n]$.

For any right G-set Y, let $\mathrm{Aut}_G(Y)$ denote the group of automorphisms of Y as a right G-set, and let Y/G denote the set of G-orbits. For any set X, we consider the cartesian product $G \times X$ of G and X to be the free right G-set with the right action of G given by $(g, x).h = (gh, x)$ for all $(g, x) \in G \times X$ and $h \in G$. The following proposition, which has been shown in the proof of [1, Proposition 6.11], plays an important role in our theory.

Proposition 1. *Let X be a finite set, and let S_X be the symmetric group on X. If $\gamma \in \mathrm{Aut}_G(G \times X)$, then there exists a unique pair $(\xi, \beta) \in G^X \times S_X$ such that $\gamma(g, x) = (\xi(x)g, \beta(x))$ for all $(g, x) \in G \times X$. Further, this correspondence from $\mathrm{Aut}_G(G \times X)$ to $G^X \times S_X$ is bijective.*

Using Proposition 1, we identify $G \wr S_n$ with $\mathrm{Aut}_G(G \times [n])$ so that an element $(f;\pi) \in G \wr S_n$ is regarded as an automorphism $\gamma_{(f;\pi)} \in \mathrm{Aut}_G(G \times [n])$ satisfying $\gamma_{(f;\pi)}(g, i) = (f(\pi(i))g, \pi(i))$ for all $(g, i) \in G \times [n]$ (see also [3, 2.11]). Further, if Y is a free right G-set and if $n = |Y/G|$, then $\mathrm{Aut}_G(Y) \cong G \wr S_n$, because, for a system of representatives Ω of Y/G, $Y \cong G \times \Omega$ as right G-sets [1, Proposition 6.11].

A right G-set Y with the left action of A given by a homomorphism from A to $\mathrm{Aut}_G(Y)$ is called an (A, G)-biset. Also, an (A, G)-biset Y is said to be finite G-free if it is finite free as a right G-set. A mapping σ between (A, G)-bisets is called a morphism of (A, G)-bisets if it is a morphism both of left A-sets and of right G-sets.

Definition 1. The category $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$ is defined as follows:

- The objects are triples (Y, σ, X), where Y is a finite G-free (A, G)-biset, X is a finite left A-set, viewed as an (A, G)-biset with the trivial right action of G, and $\sigma : Y \to X$ is a morphism of (A, G)-bisets, inducing an isomorphism of left A-sets $\overline{\sigma} : Y/G \to X$;

- A morphism $(Y, \sigma, X) \to (Y', \sigma', X')$ is a pair (γ, β), where $\gamma : Y \to Y'$ and $\beta : X \to X'$ are morphisms of (A, G)-bisets, such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\gamma"] \arrow[d, "\sigma"'] & Y' \arrow[d, "\sigma'"] \\ X \arrow[r, "\beta"'] & X' \end{tikzcd}$$

is commutative. The composition is given by $(\gamma, \beta) \circ (\gamma', \beta') := (\gamma \circ \gamma', \beta \circ \beta')$, and the identity $\mathrm{Id}_{(Y,\sigma,X)}$ is the pair $(\mathrm{Id}_Y, \mathrm{Id}_X)$ of the identities $\mathrm{Id}_Y : Y \to Y$ and $\mathrm{Id}_X : X \to X$.

For any set X, $\mathrm{Pr} : G \times X \to X$ denotes the projection. If $\varphi \in \mathrm{Hom}(A, G \wr S_n)$, then we denote by $(G \times [n])_\varphi$ the object $(G \times [n], \mathrm{Pr}, [n])$ of $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$ coming from the left action φ of A on $G \times [n]$.

Proposition 2. *The following statements hold*:

(a) *Suppose that (Y, σ, X) is an object of $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$ and that $n = |X|$. Then there exists a homomorphism $\varphi \in \mathrm{Hom}(A, G \wr S_n)$ such that $(G \times [n])_\varphi \cong (Y, \sigma, X)$.*

(b) *Let $(G \times [n])_\varphi$, $(G \times [n])_{\varphi_*}$ be a pair of objects in $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$. Then the set of isomorphisms from $(G \times [n])_\varphi$ to $(G \times [n])_{\varphi_*}$ consists of all pairs (γ, π) with $\gamma = (f; \pi) \in G \wr S_n$ such that $\varphi_*(a) = \gamma \varphi(a) \gamma^{-1}$ for all $a \in A$.*

Proof. Let Ω be a system of representatives of Y/G, and let $\gamma : G \times \Omega \to Y$ be the isomorphism of right G-sets defined by $\gamma(g, x) = xg$ for all $(g, x) \in G \times \Omega$. Let $\tau : Y \to Y/G$ be the morphism of left A-sets such that, for all $y \in Y$, $\tau(y)$ is the G-orbit containing y. We now suppose that A acts on Y via a homomorphism $\eta \in \mathrm{Hom}(A, \mathrm{Aut}_G(Y))$. Then there exists an object $(G \times \Omega, \tau \circ \gamma, Y/G)$ of $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$ coming from the left action ψ of A on $G \times \Omega$ defined by $\eta(a) \circ \gamma = \gamma \circ \psi(a)$ for all $a \in A$. Further, $(\gamma, \bar{\sigma}) : (G \times \Omega, \tau \circ \gamma, Y/G) \to (Y, \sigma, X)$ is an isomorphism and there exists a homomorphism $\varphi \in \mathrm{Hom}(A, G \wr S_n)$ such that $(G \times [n])_\varphi \cong (G \times \Omega, \tau \circ \gamma, Y/G)$, whence (a) follows. The statement (b) is straightforward. This completes the proof of Proposition 2. □

3. The Wohlfahrt series for $G \wr S_n$

The goal in this section is to give the exponential formula of the Wohlfahrt series for the wreath product $G \wr S_n$.

Given a pair of (A, G)-bisets Y_1, Y_2, their disjoint union $Y_1 \dot{\cup} Y_2$ is an (A, G)-biset. The sum of a pair of objects (Y, σ, X), (Y', σ', X') in $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$ is

$$(Y, \sigma, X) + (Y', \sigma', X') := (Y \dot{\cup} Y', \sigma \dot{\cup} \sigma', X \dot{\cup} X'),$$

where $(\sigma \dot\cup \sigma')(y) = \sigma(y)$ if $y \in Y$ and $(\sigma \dot\cup \sigma')(y) = \sigma'(y)$ if $y \in Y'$, which is a coproduct in $G\text{-Set}_{\mathrm{f}}^{A}$. For any object (Y, σ, X) of $G\text{-Set}_{\mathrm{f}}^{A}$, the left A-set X is a disjoint union of the A-orbits, say $X_1, X_2, \ldots$, and also Y is a disjoint union of finite G-free (A, G)-bisets, say $Y_1, Y_2, \ldots$, such that σ induces morphisms of (A, G)-bisets $\sigma_i : Y_i \to X_i$, $i = 1, 2, \ldots$, which yields $(Y, \sigma, X) = (Y_1, \sigma_1, X_1) + (Y_2, \sigma_2, X_2) + \cdots$. Thus an object (Y, σ, X) of $G\text{-Set}_{\mathrm{f}}^{A}$ is connected if and only if X is a non-empty transitive left A-set, or equivalently, there is a subgroup B of A of finite index such that $A/B \cong X$ as left A-sets.

Lemma 1. *Let B be a subgroup of A of index n and $T_B = \{a_1, a_2, \ldots, a_n\}$ a left transversal of B. Suppose that $\kappa \in \mathrm{Hom}(B, G)$ and that $\theta \in G^{T_B}$. Then there exists a mapping $\varphi_{(T_B,\kappa,\theta)} : A \to \mathrm{Aut}_G(G \times (A/B))$ such that*

$$\varphi_{(T_B,\kappa,\theta)}(a)(g, a_j B) = \bigl(\theta(a_{j'})\kappa\bigl(a_{j'}^{-1} a a_j\bigr)\theta(a_j)^{-1} g, a_{j'} B\bigr), \quad \text{where } a a_j B = a_{j'} B,$$

for all $a \in A$, $g \in G$, and $j \in [n]$. Further, it is a homomorphism.

Proof. The first assertion follows from Proposition 1. It is easy to see that $\varphi_{(T_B,\kappa,\theta)}$ is a homomorphism. □

Definition 2. Under the notation of Lemma 1, we denote by $(G \times [T_B])_{(\kappa,\theta)}$ the object $(G \times (A/B), \mathrm{Pr}, A/B)$ coming from the left action $\varphi_{(T_B,\kappa,\theta)}$ of A on $G \times (A/B)$.

Let ε_A and ε_G be the identities of A and G, respectively. The following proposition enables us to determine the isomorphism classes of connected objects of $G\text{-Set}_{\mathrm{f}}^{A}$.

Proposition 3. *Let B be a subgroup of A of finite index and T_B a left transversal of B containing ε_A. Suppose that $\varphi \in \mathrm{Hom}(A, \mathrm{Aut}_G(G \times (A/B)))$ and that, for each element a of A, there is a mapping $\xi_a \in G^{A/B}$ with $\varphi(a)(g, xB) = (\xi_a(xB)g, axB)$ for all $(g, xB) \in G \times (A/B)$. Then there is a unique pair $(\kappa, \theta) \in \mathrm{Hom}(B, G) \times G^{T_B}$ with $\theta(\varepsilon_A) = \varepsilon_G$ such that $(G \times [T_B])_{(\kappa,\theta)}$ expresses the object $(G \times (A/B), \mathrm{Pr}, A/B)$ of $G\text{-Set}_{\mathrm{f}}^{A}$ coming from the left action φ of A on $G \times (A/B)$.*

Proof. Suppose that A acts on $G \times (A/B)$ via φ and that $T_B = \{a_1, a_2, \ldots, a_n\}$ with $a_1 = \varepsilon_A$. Define $(\kappa, \theta) \in \mathrm{Hom}(B, G) \times G^{T_B}$ by $\kappa(b) = \xi_b(B)$ for all $b \in B$ and $\theta(a_j) = \xi_{a_j}(B)$ for all $j \in [n]$. Then $b.(\varepsilon_G, B) = (\kappa(b), B)$ for all $b \in B$ and $a_j.(\varepsilon_G, B) = (\theta(a_j), a_j B)$ for all $j \in [n]$. Further, for any $a \in A$, $g \in G$, and $j \in [n]$, if $a a_j B = a_{j'} B$, then

$$\begin{aligned} a.(g, a_j B) &= (a a_j).\bigl(\theta(a_j)^{-1} g, B\bigr) = a_{j'}.\bigl(\kappa\bigl(a_{j'}^{-1} a a_j\bigr)\theta(a_j)^{-1} g, B\bigr) \\ &= \bigl(\theta(a_{j'})\kappa\bigl(a_{j'}^{-1} a a_j\bigr)\theta(a_j)^{-1} g, a_{j'} B\bigr). \end{aligned}$$

Thus $(G \times [T_B])_{(\kappa,\theta)} = (G \times (A/B), \mathrm{Pr}, A/B)$. Also, $\theta(\varepsilon_A) = \varepsilon_G$ and the uniqueness of (κ, θ) is clear. We have thus proved the proposition. □

We require the following proposition.

Proposition 4. *For each subgroup B of A of finite index, we fix a left transversal T_B of B containing ε_A. Suppose that B is a subgroup of A of finite index and that $(\kappa, \theta) \in \mathrm{Hom}(B, G) \times G^{T_B}$ with $\theta(\varepsilon_A) = \varepsilon_G$. We denote by $\mathscr{I}_{(T_B,\kappa,\theta)}$ the set of all pairs $((G \times [T_{B_*}])_{(\kappa_*,\theta_*)}, (\gamma, \beta))$, where $(\kappa_*, \theta_*) \in \mathrm{Hom}(B_*, G) \times G^{T_{B_*}}$ with $\theta_*(\varepsilon_A) = \varepsilon_G$ and (γ, β) is an isomorphism from $(G \times [T_B])_{(\kappa,\theta)}$ to $(G \times [T_{B_*}])_{(\kappa_*,\theta_*)}$. Then there exists a bijection from $\mathscr{I}_{(T_B,\kappa,\theta)}$ to $G^{A/B} \times T_B$.*

Proof. Suppose that $T_B = \{a_1, a_2, \ldots, a_n\}$. Let $((G \times [T_{B_*}])_{(\kappa_*,\theta_*)}, (\gamma, \beta)) \in \mathscr{I}_{(T_B,\kappa,\theta)}$. Then $\beta : A/B \to A/B_*$ is an isomorphism of left A-sets. Take an element a of A so that $\beta(B) = aB_*$. Then $aB_* = \beta(B) = b\beta(B) = baB_*$ for all $b \in B$, and hence B^a is a subgroup of B_*. Likewise, B_* is conjugate to a subgroup of B, and consequently, $B_* = B^a$. We can now choose a unique element a_i of T_B so that $\beta(B) = Ba_i^{-1}$ and $B_* = B^{a_i^{-1}}$. Further, since γ is a morphism of free right G-sets, these exists a unique mapping $\xi \in G^{A/B}$ such that $\gamma(g, a_jB) = (\xi(a_jB)g, \beta(a_jB)) = (\xi(a_jB)g, a_ja_i^{-1}B_*)$ for all $g \in G$ and $j \in [n]$, which defines an injection from $\mathscr{I}_{(T_B,\kappa,\theta)}$ to $G^{A/B} \times T_B$.

Conversely, let $(\xi, a_i) \in G^{A/B} \times T_B$. Define $B_* = B^{a_i^{-1}}$, and let $\beta : A/B \to A/B_*$ be the morphism of left A-sets defined by $\beta(a_jB) = a_ja_i^{-1}B_* = a_jBa_i^{-1}$ for all $j \in [n]$. Then there exists an isomorphism $\gamma : G \times (A/B) \to G \times (A/B_*)$ of right G-sets satisfying $\gamma(g, a_jB) = (\xi(a_jB)g, \beta(a_jB))$ for all $g \in G$ and $j \in [n]$. Hence, by Proposition 3, there is a unique pair $(\kappa_*, \theta_*) \in \mathrm{Hom}(B_*, G) \times G^{T_{B_*}}$ with $\theta_*(\varepsilon_A) = \varepsilon_G$ such that $(G \times [T_{B_*}])_{(\kappa_*,\theta_*)}$ expresses an object $(G \times (A/B_*), \mathrm{Pr}, A/B_*)$ of $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$ coming from the left action φ of A on $G \times (A/B_*)$ defined by $\varphi(a) \circ \gamma = \gamma \circ \varphi_{(T_B,\kappa,\theta)}(a)$ for all $a \in A$, where $\varphi_{(T_B,\kappa,\theta)}$ is defined in Lemma 1. Now (γ, β) is an isomorphism from $(G \times [T_B])_{(\kappa,\theta)}$ to $(G \times [T_{B_*}])_{(\kappa_*,\theta_*)}$, and thereby the injection from $\mathscr{I}_{(T_B,\kappa,\theta)}$ to $G^{A/B} \times T_B$ defined in the preceding paragraph is bijective. This completes the proof of Proposition 4. □

It is clear that the category $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$ is a skeletally small and locally finite KS category. We are now in a position to apply [10, 5.8. Theorem] to $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$.

Proposition 5. *For each subgroup B of A of finite index, we fix a left transversal T_B of B containing ε_A, and define $\mathscr{L}(T_B, G) := \mathrm{Hom}(B, G) \times \{\theta \in G^{T_B} \mid \theta(\varepsilon_A) = \varepsilon_G\}$. Let r be a mapping from the set of objects of $G\text{-}\mathrm{Set}_{\mathrm{f}}^A$ to a $\mathbb{Q}$-algebra satisfying the following conditions*:

(i) $r(Y, \sigma, X) = r(Y', \sigma', X')$ *if* $(Y, \sigma, X) \cong (Y', \sigma', X')$;
(ii) $r(\emptyset) = 1$, $r((Y_1, \sigma_1, X_1) + (Y_2, \sigma_2, X_2)) = r(Y_1, \sigma_1, X_1) r(Y_2, \sigma_2, X_2)$.

Then

$$\sum_{n=0}^{\infty} \sum_{\varphi \in \mathrm{Hom}(A, G \wr S_n)} \frac{r_\varphi}{|G|^n n!} t^n = \exp\left(\sum_{B \leqslant_f A} \sum_{(\kappa,\theta) \in \mathscr{L}(T_B,G)} \frac{r_{(T_B,\kappa,\theta)}}{|G|^{|A:B|} |A:B|} t^{|A:B|} \right),$$

where $r_\varphi = r((G \times [n])_\varphi)$ and $r_{(T_B,\kappa,\theta)} = r((G \times [T_B])_{(\kappa,\theta)})$.

Proof. Let $\mathrm{Con}(G\text{-}\mathrm{Set}_{\mathrm{f}}^{A})$ be the full subcategory of connected objects of $G\text{-}\mathrm{Set}_{\mathrm{f}}^{A}$. By [10, 5.3], we can substitute $t^{(Y,\sigma,X)}$ for $r(Y,\sigma,X)t^{|X|}$ on both sides of the equation in [10, 5.8. Theorem] with $\mathscr{E} = G\text{-}\mathrm{Set}_{\mathrm{f}}^{A}$. Hence

$$\sum'_{Z=(Y,\sigma,X)\in G\text{-}\mathrm{Set}_{\mathrm{f}}^{A}} \frac{r(Z)}{|\operatorname{Aut}(Z)|} t^{|X|} = \exp\Biggl(\sum'_{Z=(Y,\sigma,X)\in \mathrm{Con}(G\text{-}\mathrm{Set}_{\mathrm{f}}^{A})} \frac{r(Z)}{|\operatorname{Aut}(Z)|} t^{|X|}\Biggr).$$

The proposition now follows from Lemma 1 and Propositions 2–4. □

Corollary 1 [4,8]. *We have*

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, G \wr S_n)|}{|G|^n n!} t^n = \exp\Biggl(\sum_{B \leqslant_f A} \frac{|\operatorname{Hom}(B, G)|}{|G|\,|A:B|} t^{|A:B|}\Biggr).$$

Proof. The assertion follows from Proposition 5 with the mapping r defined by $r(Y,\sigma,X) = 1$ for all objects (Y,σ,X) of $G\text{-}\mathrm{Set}_{\mathrm{f}}^{A}$. □

4. Even permutation representations

A Wohlfahrt series is expressed in the form $\sum_{n=0}^{\infty} |\operatorname{Hom}(A, K_n)| t^n / |G|^n n!$; by substituting the variable t of this series for $|G|t$, we obtain the original series. In this section we establish a fundamental theorem for enumerating homomorphisms from A to $G \wr S_n$, and present the exponential formula of the Wohlfahrt series for $G \wr A_n$. Recall that $\mathscr{C}_p(A)$ is the set of minimal subgroups of $A/\Phi_p(A)$ and that each element of $\mathscr{C}_p(A)$ is denoted by $\langle \bar{c} \rangle$ for an element $c \in A - \Phi_p(A)$ with $c^p \in \Phi_p(A)$, where $\bar{c} = c\Phi_p(A)$. The following theorem relates to [5, Theorem 3.1].

Theorem 1. *Suppose that homomorphisms $\zeta_n \in \operatorname{Hom}(S_n, \langle\omega\rangle)$, $n = 1, 2, \ldots,$ satisfy the condition that either $\operatorname{Ker}\zeta_n = S_n$ for any n, or $p = 2$ and $\operatorname{Ker}\zeta_n = A_n$ for any n. Let χ be a homomorphism form G to $\langle\omega\rangle$, and let $\chi_1, \chi_2, \ldots$ be the sequence of homomorphisms $\chi_n \in \operatorname{Hom}(G \wr S_n, \langle\omega\rangle)$, $n = 1, 2, \ldots,$ defined by*

$$\chi_n(f;\pi) = \chi\bigl(f(1)\bigr)\chi\bigl(f(2)\bigr)\cdots\chi\bigl(f(n)\bigr)\zeta_n(\pi)$$

for all $(f;\pi) \in G \wr S_n$. Set $K_n = \operatorname{Ker}\chi_n$. Then

$$\bigl|A : \Phi_p(A)\bigr| \sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, K_n)|}{|G|^n n!} t^n$$
$$= \exp\Biggl(\sum_{B \leqslant_f A} \frac{|\operatorname{Hom}(B, G)|}{|G|\,|A:B|} t^{|A:B|}\Biggr)$$

$$+\sum_{\langle\bar{c}\rangle\in\mathscr{C}_p(A)}\sum_{i=1}^{p-1}\exp\left(\sum_{B\leqslant_f A}\sum_{\kappa\in\mathrm{Hom}(B,G)}\frac{\zeta_B(c)\cdot\overline{\chi\circ\kappa}(V_{B/\Phi_p(B)}(c))^i}{|G|\,|A:B|}t^{|A:B|}\right),$$

where $\overline{\chi\circ\kappa}\in\mathrm{Hom}(B/\Phi_p(B),\langle\omega\rangle)$ *is the homomorphism defined by*

$$\overline{\chi\circ\kappa}\big(b\Phi_p(B)\big)=\chi\big(\kappa(b)\big)$$

for all $b\in B$*, and* $\zeta_B(c)=1$ *if* $\mathrm{Ker}\,\zeta_n=S_n$ *for any* n *and* $\zeta_B(c)=\mathrm{sgn}_B(c)$ *if* $p=2$ *and* $\mathrm{Ker}\,\zeta_n=A_n$ *for any* n*. Here* $\zeta_B(c)$ *and* $V_{B/\Phi_p(B)}(c)$ *are independent of the choice of an element* c *in a coset* $\langle\bar{c}\rangle\in\mathscr{C}_p(A)$*.*

Proof. Suppose that $\varphi\in\mathrm{Hom}(A,G\wr S_n)$. Then either $|A:\mathrm{Ker}(\chi_n\circ\varphi)|=p$ or $A=\mathrm{Ker}(\chi_n\circ\varphi)$, and further, $\mathrm{Ker}(\chi_n\circ\varphi)$ contains $\Phi_p(A)$. If $|A:\mathrm{Ker}(\chi_n\circ\varphi)|=p$, then

$$\begin{aligned}\sum_{\langle\bar{c}\rangle\in\mathscr{C}_p(A)}\sum_{i=1}^{p-1}\chi_n\big(\varphi(c)\big)^i&=(p-1)\sharp\big\{\langle\bar{c}\rangle\in\mathscr{C}_p(A)\;\big|\;c\in\mathrm{Ker}(\chi_n\circ\varphi)\big\}\\&\qquad-\sharp\big\{\langle\bar{c}\rangle\in\mathscr{C}_p(A)\;\big|\;c\notin\mathrm{Ker}(\chi_n\circ\varphi)\big\}\\&=-1,\end{aligned}$$

because $\sum_{i=1}^{p-1}\omega^i=-1$ and $A/\Phi_p(A)$ is an elementary abelian p-group. Note that the number of subgroups of order p in an elementary abelian p-group of order p^s is equal to $(p^s-1)/(p-1)$. Hence we obtain

$$1+\sum_{\langle\bar{c}\rangle\in\mathscr{C}_p(A)}\sum_{i=1}^{p-1}\chi_n\big(\varphi(c)\big)^i=\begin{cases}\big|A:\Phi_p(A)\big|, & \text{if } A=\mathrm{Ker}(\chi_n\circ\varphi),\\ 0, & \text{otherwise.}\end{cases}$$

Consequently, since $\mathrm{Hom}(A,K_n)=\{\varphi\in\mathrm{Hom}(A,G\wr S_n)\mid A=\mathrm{Ker}(\chi_n\circ\varphi)\}$, it follows that

$$\begin{aligned}&\big|A:\Phi_p(A)\big|\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}(A,K_n)|}{|G|^n n!}t^n\\&\quad=\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}(A,G\wr S_n)|}{|G|^n n!}t^n+\sum_{\langle\bar{c}\rangle\in\mathscr{C}_p(A)}\sum_{i=1}^{p-1}\sum_{n=0}^{\infty}\sum_{\varphi\in\mathrm{Hom}(A,G\wr S_n)}\frac{\chi_n(\varphi(c))^i}{|G|^n n!}t^n.\qquad(2)\end{aligned}$$

Suppose that $c\in A$. Using Proposition 2, we define a mapping r from the set of objects of G-$\mathrm{Set}_{\mathrm{f}}^A$ to the complex numbers by setting $r(Y,\sigma,X)=\chi_n(\varphi(c))$ if $(Y,\sigma,X)\cong(G\times[n])_\varphi$, and $r(\emptyset)=1$. Note that, if objects $(G\times[n])_\varphi$ and $(G\times[n])_{\varphi_*}$ are isomorphic in G-$\mathrm{Set}_{\mathrm{f}}^A$, then $\chi_n(\varphi(c))=\chi_n(\varphi_*(c))$ by Proposition 2(b). Further, given a pair of objects

$(G \times [n_1])_{\varphi_1}$, $(G \times [n_2])_{\varphi_2}$ in $G\text{-Set}_{\mathrm{f}}^A$ with $n = n_1 + n_2$, there exists a homomorphism $\varphi \in \mathrm{Hom}(A, G \wr S_n)$ such that

$$\big(G \times [n]\big)_{\varphi} \cong \big(G \times [n_1]\big)_{\varphi_1} + \big(G \times [n_2]\big)_{\varphi_2}$$

and

$$\chi_n\big(\varphi(c)\big) = \chi_{n_1}\big(\varphi_1(c)\big)\chi_{n_2}\big(\varphi_2(c)\big).$$

Thus the mapping r satisfies the conditions (i) and (ii) in Proposition 5. Also, under the notation of Proposition 5, if B is a subgroup of A of finite index and if $(\kappa, \theta) \in \mathscr{L}(T_B, G)$, then $r((G \times [T_B])_{(\kappa,\theta)}) = \zeta_B(c) \cdot \overline{\chi \circ \kappa}(V_{B/\Phi_p(B)}(c))$, which is independent of the choice of θ. It now follows from Proposition 5 that

$$\sum_{n=0}^{\infty} \sum_{\varphi \in \mathrm{Hom}(A, G \wr S_n)} \frac{\chi_n(\varphi(c))}{|G|^n n!} t^n = \exp\left(\sum_{B \leqslant_f A} \sum_{\kappa \in \mathrm{Hom}(B,G)} \frac{\zeta_B(c) \cdot \overline{\chi \circ \kappa}(V_{B/\Phi_p(B)}(c))}{|G|\,|A:B|} t^{|A:B|} \right).$$

This formula, together with Corollary 1, enables us to obtain the desired result as a consequence of Eq. (2). We have thus proved the theorem. □

We classify the group K_n in Theorem 1, according as $\operatorname{Ker}\zeta_n = S_n$ for any n or $p = 2$ and $\operatorname{Ker}\zeta_n = A_n$ for any n, and $\operatorname{Ker}\chi = G$ or $\operatorname{Ker}\chi \neq G$.

Case 1. $\operatorname{Ker}\zeta_n = S_n$, $\operatorname{Ker}\chi = G$, and $K_n = G \wr S_n$.

Case 2. $p = 2$, $\operatorname{Ker}\zeta_n = A_n$, $\operatorname{Ker}\chi = G$, and $K_n = G \wr A_n$.

Case 3. $\operatorname{Ker}\zeta_n = S_n$, $\operatorname{Ker}\chi \neq G$, and

$$K_n = \big\{(f;\pi) \in G \wr S_n \;\big|\; \chi\big(f(1)f(2)\cdots f(n)\big) = 1\big\}.$$

Case 4. $p = 2$, $\operatorname{Ker}\zeta_n = A_n$, $\operatorname{Ker}\chi \neq G$, and

$$K_n = \big\{(f;\pi) \in G \wr S_n \;\big|\; \chi\big(f(1)f(2)\cdots f(n)\big)\mathrm{sgn}(\pi) = 1\big\},$$

where sgn is the usual sign.

The assertion of Theorem 1 in Case 1 is Corollary 1, and the one in Case 2 is the following corollary to Theorem 1.

Corollary 2. *We have*

$$\begin{aligned}\bigl|A:\Phi_2(A)\bigr|\sum_{n=0}^{\infty}\frac{|\operatorname{Hom}(A,G\wr A_n)|}{|G|^n n!}\,t^n &= \exp\biggl(\sum_{B\leqslant_f A}\frac{|\operatorname{Hom}(B,G)|}{|G|\,|A:B|}\,t^{|A:B|}\biggr)\\ &\quad+\sum_{\langle\bar c\rangle\in\mathscr{C}_2(A)}\exp\biggl(\sum_{B\leqslant_f A}\frac{\operatorname{sgn}_B(c)\cdot|\operatorname{Hom}(B,G)|}{|G|\,|A:B|}\,t^{|A:B|}\biggr).\end{aligned}$$

In particular,

$$\begin{aligned}\bigl|A:\Phi_2(A)\bigr|\sum_{n=0}^{\infty}\frac{|\operatorname{Hom}(A,A_n)|}{n!}\,t^n = \exp\biggl(\sum_{B\leqslant_f A}\frac{1}{|A:B|}\,t^{|A:B|}\biggr)\\ +\sum_{\langle\bar c\rangle\in\mathscr{C}_2(A)}\exp\biggl(\sum_{B\leqslant_f A}\frac{\operatorname{sgn}_B(c)}{|A:B|}\,t^{|A:B|}\biggr).\end{aligned}$$

Proof. The corollary is an immediate consequence of Theorem 1. □

Remark. If A is abelian and if B is a subgroup of A of finite index, then, for each $a\in A$, $\operatorname{sgn}_B(a)=1$ if and only if either $\langle aB\rangle$ does not include any Sylow 2-subgroup of A/B or else A/B is of odd order [8, Lemmas 2.1]. (The first statement of [8, Lemma 4.1] is missing in the case where A/B is of odd order.) The second assertion of Corollary 2 is now equivalent to [8, Theorem 1.1] if A is abelian, and is equivalent to the fact in [7, Chapter 4, Problem 22] if A is a finite cyclic group. (The formula [10, (6.5.d)] is not correct. However, the idea in [10, 6.5] is useful for the proof of Theorem 1.)

5. Various Wohlfahrt series

We devote the rest of this paper to the applications of Theorem 1 to Cases 3 and 4. The formula of Theorem 1 in the case where $\operatorname{Ker}\chi\neq G$ seems to be so implicit that we try to give a slightly explicit formula under a certain additional condition.

Lemma 2. *Suppose that $\chi\in\operatorname{Hom}(G,\langle\omega\rangle)$ and that $c\in A$. If either A or G is abelian, then the number of homomorphisms $\psi\in\operatorname{Hom}(A,G)$ satisfying $\chi(\psi(c))=\omega^j$ is independent of the choice of an integer j with $1\leqslant j\leqslant p-1$.*

Proof. If G is abelian, then we can identify $\operatorname{Hom}(A,G)$ with $\operatorname{Hom}(A/A',G)$, where A' is the commutator subgroup of A. Hence we may assume that A is abelian. Let K be the intersection of all kernels of homomorphisms $\psi\in\operatorname{Hom}(A,G)$. Then K is a subgroup of A of finite index, and hence A/K is a finite abelian group. Now, since $\operatorname{Hom}(A,G)$ is identified with $\mathrm{Hom}(A/K, G)$, we may assume that A is a finite abelian p-group. For each integer i, we define

$$\mathrm{Hom}\big(A, G; c, \omega^i\big) = \big\{\psi \in \mathrm{Hom}(A, G) \bigm| \chi\big(\psi(c)\big) = \omega^i\big\}.$$

Let i and j be arbitrary positive integers less than p, and let ℓ be a positive integer satisfying $i\ell \equiv j \bmod p$. If $\psi \in \mathrm{Hom}(A, G; c, \omega^i)$, then a homomorphism $\psi^{(\ell)} \in \mathrm{Hom}(A, G)$ is defined by setting $\psi^{(\ell)}(a) = \psi(a)^\ell$ for all $a \in A$, because A is abelian. Here we get $\chi(\psi^{(\ell)}(c)) = \chi(\psi(c))^\ell = \omega^{i\ell} = \omega^j$. Hence there is a correspondence

$$\lambda_{i,j} : \mathrm{Hom}\big(A, G; c, \omega^i\big) \ni \psi \longrightarrow \psi^{(\ell)} \in \mathrm{Hom}\big(A, G; c, \omega^j\big).$$

Let s be a positive integer satisfying $\ell s \equiv 1 \bmod |A|$. Suppose that $\psi_1^{(\ell)} = \psi_2^{(\ell)}$ with $\psi_1, \psi_2 \in \mathrm{Hom}(A, G; c, \omega^i)$. Then we obtain

$$\psi_1(a) = \psi_1(a)^{\ell s} = \psi_2(a)^{\ell s} = \psi_2(a)$$

for all $a \in A$, whence $\psi_1 = \psi_2$. Thus the correspondence $\lambda_{i,j}$ is one-to-one. Since i and j are arbitrary, we now conclude that $\sharp\,\mathrm{Hom}(A, G; c, \omega^i) = \sharp\,\mathrm{Hom}(A, G; c, \omega^j)$. This completes the proof of Lemma 2. □

Definition 3. Suppose that $\chi \in \mathrm{Hom}(G, \langle\omega\rangle)$ and that B is a subgroup of A of finite index. For each element c of A, define

$$\begin{aligned}\ell_c(B; \chi) := {}&\sharp\big\{\kappa \in \mathrm{Hom}(B, G) \bigm| V_{B/\Phi_p(B)}(c) \in \mathrm{Ker}\,\overline{\chi \circ \kappa}\big\}\\ &- \frac{1}{p-1}\sharp\big\{\kappa \in \mathrm{Hom}(B, G) \bigm| V_{B/\Phi_p(B)}(c) \notin \mathrm{Ker}\,\overline{\chi \circ \kappa}\big\}.\end{aligned}$$

Here $\overline{\chi \circ \kappa}$ is defined in Theorem 1.

We can now show a formula of the Wohlfahrt series in Case 3.

Theorem 2. *Suppose that $\chi \in \mathrm{Hom}(G, \langle\omega\rangle)$. Let K_n be the subgroup of $G \wr S_n$ consisting of all elements $(f; \pi)$ satisfying $\chi(f(1)f(2)\cdots f(n)) = 1$. If either A or G is abelian, then*

$$\begin{aligned}\big|A : \Phi_p(A)\big| \sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, K_n)|}{|G|^n n!} t^n = {}&\exp\Bigg(\sum_{B \leqslant_f A} \frac{|\mathrm{Hom}(B, G)|}{|G|\,|A : B|} t^{|A:B|}\Bigg)\\ &+ (p-1) \sum_{\langle\bar{c}\rangle \in \mathscr{C}_p(A)} \exp\Bigg(\sum_{B \leqslant_f A} \frac{\ell_c(B; \chi)}{|G|\,|A : B|} t^{|A:B|}\Bigg).\end{aligned}$$

Here $\ell_c(B; \chi)$ is independent of the choice of an element c in a coset $\langle\bar{c}\rangle \in \mathscr{C}_p(A)$.

Proof. If B is a subgroup of A of finite index and if $\langle\bar{c}\rangle \in \mathscr{C}_p(A)$, then the number of homomorphisms $\kappa \in \mathrm{Hom}(B, G)$ satisfying $\overline{\chi\circ\kappa}(V_{B/\Phi_p(B)}(c)) = \omega^j$ is independent of the choice of an integer j with $1 \leqslant j \leqslant p-1$ by Lemma 2, and hence

$$\ell_c(B;\chi) = \sum_{\kappa\in\mathrm{Hom}(B,G)} \overline{\chi\circ\kappa}\big(V_{B/\Phi_p(B)}(c)\big)^i$$

for any integer i with $1 \leqslant i \leqslant p-1$. The assertion now follows from Theorem 1. $\square$

The second assertion of the following corollary is equivalent to [8, Theorem 1.2] if A is abelian.

Corollary 3. *Let K_n be the subgroup of $\langle\omega\rangle \wr S_n$ consisting of all elements $(f;\pi)$ satisfying $f(1)f(2)\cdots f(n) = 1$. Then*

$$\begin{aligned}
&\big|A:\Phi_p(A)\big| \sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A,K_n)|}{p^n n!}\, t^n \\
&\quad = \exp\bigg(\sum_{B\leqslant_f A} \frac{|B:\Phi_p(B)|}{p|A:B|}\, t^{|A:B|}\bigg) \\
&\qquad + (p-1) \sum_{\langle\bar{c}\rangle\in\mathscr{C}_p(A)} \exp\bigg(\sum_{\substack{B\leqslant_f A\\ c\in\mathrm{Ker}\, V_{B/\Phi_p(B)}}} \frac{|B:\Phi_p(B)|}{p|A:B|}\, t^{|A:B|}\bigg),
\end{aligned}$$

where the summation $\sum_{B\leqslant_f A,\, c\in\mathrm{Ker}\, V_{B/\Phi_p(B)}}$ is over all subgroups B of A of finite index such that $c \in \mathrm{Ker}\, V_{B/\Phi_p(B)}$. In particular,

$$\begin{aligned}
&\big|A:\Phi_2(A)\big| \sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A,W(D_n))|}{2^n n!}\, t^n \\
&\quad = \exp\bigg(\sum_{B\leqslant_f A} \frac{|B:\Phi_2(B)|}{2|A:B|}\, t^{|A:B|}\bigg) \\
&\qquad + \sum_{\langle\bar{c}\rangle\in\mathscr{C}_2(A)} \exp\bigg(\sum_{\substack{B\leqslant_f A\\ c\in\mathrm{Ker}\, V_{B/\Phi_2(B)}}} \frac{|B:\Phi_2(B)|}{2|A:B|}\, t^{|A:B|}\bigg).
\end{aligned}$$

Proof. The assertion is a consequence of Theorem 2 and Lemma 3 below. $\square$

Lemma 3. *Let B be a subgroup of A of finite index. Then*

$$\big|\mathrm{Hom}\big(B,\langle\omega\rangle\big)\big| = \big|B:\Phi_p(B)\big|.$$

Further, for any automorphism χ of $\langle\omega\rangle$ and for any coset $\langle\bar{c}\rangle \in \mathscr{C}_p(A)$,

$$\ell_c(B;\chi) = \begin{cases} \bigl|B : \Phi_p(B)\bigr|, & \textit{if } c \in \operatorname{Ker} V_{B/\Phi_p(B)}, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. It is easy to see that $|\operatorname{Hom}(B/\Phi_p(B), \langle\omega\rangle)| = |B : \Phi_p(B)|$. Also, there is a natural bijection between $\operatorname{Hom}(B, \langle\omega\rangle)$ and $\operatorname{Hom}(B/\Phi_p(B), \langle\omega\rangle)$. Hence we have $|\operatorname{Hom}(B, \langle\omega\rangle)| = |B : \Phi_p(B)|$. If $c \in \operatorname{Ker} V_{B/\Phi_p(B)}$, then $\ell_c(B;\chi) = |B : \Phi_p(B)|$. So we assume that $c \notin \operatorname{Ker} V_{B/\Phi_p(B)}$. Then, since $\Phi_p(A) \leqslant \operatorname{Ker} V_{B/\Phi_p(B)}$, we have $|\langle V_{B/\Phi_p(B)}(c)\rangle| = |\langle\bar{c}\rangle| = p$. The assumption that $\operatorname{Ker}\chi = \{1\}$ now yields

$$\begin{aligned} &\sharp\bigl\{\kappa \in \operatorname{Hom}\bigl(B, \langle\omega\rangle\bigr) \bigm| V_{B/\Phi_p(B)}(c) \in \operatorname{Ker} \overline{\chi \circ \kappa}\bigr\} \\ &\quad = \sharp\bigl\{\bar{\kappa} \in \operatorname{Hom}\bigl(B/\Phi_p(B), \langle\omega\rangle\bigr) \bigm| \bigl\langle V_{B/\Phi_p(B)}(c)\bigr\rangle \leqslant \operatorname{Ker}\bar{\kappa}\bigr\} \\ &\quad = \bigl|B/\Phi_p(B) : \bigl\langle V_{B/\Phi_p(B)}(c)\bigr\rangle\bigr| \\ &\quad = \bigl|B : \Phi_p(B)\bigr|/p. \end{aligned}$$

Consequently, we obtain $\ell_c(B;\chi) = 0$. This completes the proof of Lemma 3. □

We finish by stating a result in Case 4.

Theorem 3. *Let K_n be a subgroup of $\langle -1\rangle \wr S_n$ consisting of all elements $(f;\pi)$ satisfying $f(1)f(2)\cdots f(n) = \operatorname{sgn}(\pi)$. Then*

$$\begin{aligned} &\bigl|A : \Phi_2(A)\bigr| \sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, K_n)|}{2^n n!} t^n \\ &\quad = \exp\Biggl(\sum_{B \leqslant_f A} \frac{|B : \Phi_2(B)|}{2|A : B|} t^{|A:B|}\Biggr) \\ &\qquad + \sum_{\langle\bar{c}\rangle \in \mathscr{C}_2(A)} \exp\Biggl(\sum_{\substack{B \leqslant_f A \\ c \in \operatorname{Ker} V_{B/\Phi_2(B)}}} \frac{\operatorname{sgn}_B(c) \cdot |B : \Phi_2(B)|}{2|A : B|} t^{|A:B|}\Biggr). \end{aligned}$$

Proof. The theorem follows from Theorem 1 and Lemma 3. □

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